Hhh. 1068

# EXERCITATIO GEOMETRICA

DE

## *DIMENSIONE FIGURARUM*

SIVE

*Specimen Methodi Generalis.*

Dimetiendi

## QUASVIS FIGURAS

Auctore

DAVIDE GREGORIO

In Academiâ *Edinburgensi* Matheseos Professore.

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# GEORGIO

COMITI DE ABERDEIN

Vicecomiti DE FRUMARTIN

Domino *HADDO, METHLICK, TARVES & KELLIE*

# SUMMO SCOTIÆ CANCELLARIO.

Opellam hanc

*Gratitudinis & Obsequii ergo*

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# EXERCITATIO GEOMETRICA
## De
# DIMENSIONE FIGURARUM.

POstquam Nobiliſſimum de Circulo quadrando problema per aliquot ſæcula præſtantiſſimorum Geometrarum ingenia exercuiſſet, ejus examen nuper aggreſſus eſt JACOBUS GREGORIUS Patruus meus, Libro *Patavii* edito Anno 1667. quem inſcripſit *Veram Circuli & Hyperbolæ Quadraturam, &c.* ubi oſtendit Circuli, Hyperbolæ, aut Ellipſeos Sectorem quemvis, eſſe terminationem Seriei cujuſdam convergentis, cujus primi duo termini ſunt Triangulum Sectori circuli quidem aut ellipſeos inſcriptum, hyperbolæ vero circumſcriptum A, & Trapezium eidem Sectori circulari vel elliptico circumſcriptum, hyperbolico autem inſcriptum B; ſequentes duo $\sqrt{AB}$, $\frac{2AB}{A+\sqrt{AB}}$: & tertii duo eodem modo geniti ex ſecundis quo ſecundi ex primis, & ſic in infinitum. Ibidemque docuit quomodo ſerierum convergentium terminationes quando id fieri poteſt inveniantur. Ut & terminationem Seriei quæ eſſet ad Circuli, Hyperbolæ, vel Ellipſeos Sectorem non poſſe componi analyticè ex duobus primis terminis: id eſt non dari quantitatem Sectori propoſito

 æqualem,

æqualem, quæ ex ſupradicti Trianguli & Trapezii (quæ duo problema determinant) additione, ſubſtractione, multiplicatione, diviſione, radicum extractione, vel aliâ quâvis in Geometriam receptâ Operatione, harumve quâlibet Combinatione oriri poſſit. At quomodo Approxmationes ad harum figurarum areas in quotvis figurarum loca poſſint extendi, theorematis & exemplis ſatis oſtendit: ſpeciatim vero ex hyperbolæ dimenſione adjunctâ methodo conſtruendi Logarithmos facili et expeditâ. Hanc ejus primam Methodum & vere Geometricam, & quâvis ante id tempus propoſitâ generaliorem & ſimpliciorem eſſe ſatis liquet.

Proximo deinceps anno, ſuam in lucem edidit Logarithmotechniam Clariſſimus Geometra *Nicolaus Mercator*, in quâ adducit ad Hyperbolæ Quadraturam, Seriem infinito terminorum numero conſtantem, at diverſæ a *Gregorianis* Naturæ; in quâ non ultimus terminus ſed ſumma omnium dat quæſitum. Hanc Hyperbolæ dimenſionem admodum ingenioſam & calculo aptam, *Jac. Gregorius* eodem anno demonſtratione apagogicâ confirmavit. Verum ad ſimilem circuli aliarumque figurarum dimenſionem adhuc deſiderabatur methodus, ut pote quæ à reductione diviſionis imperfectæ ad infinitam ſeriem non dependeret. Huic rei poſtea intentus, multa de figurarum dimenſione aliiſque problematis per infinitas ſeries ſolvendis meditatus eſt, &



prout

prout ferebat occasio quibusdam communicavit.

Circa initium anni 1670, vir Doctissimus & ad Geometriam promovendam natus D. *Joannes Collinseus*, literis ei significavit Clarissimum virum D. *Isaacum Neutonum* methodi quadraturarum generalis compotem esse, quæ *Mercatoris* Hyperbolæ quadraturæ analoga esset; simulque seriem unicam Zonæ circulari accomodatam transmisit: videlicet si detur radius R, & latitudo Zonæ B, erit Zona =

$$2RB - \frac{B^3}{3R} - \frac{B^5}{20R^3} - \frac{B^7}{56R^5} - \frac{5B^9}{1376R^7} - \text{ &c.}$$

quam suis congruere, ac ex prius communicatis de inveniendo numero ex dato Logarithmo; vel radicem potestatis cujusvis puræ in infinitam seriem permutando, apertè consequi, eidem D. *Collinseo* postea circa mensem *Decembris* insequentem rescripsit.

Interea harum serierum infinitarum doctrinam variis problematis ex occasione applicuit, Geometriam illius ope multò auctiorem factam esse asserens, cum multis de causis, tum hâc etiam, quòd in quâvis æquatione literali, facile esset seriem infinitam ejusdem radici accommodatam exhibere, et quod unica cuivis gradui sufficeret; quodque harum tabulæ radicibus expedite inveniendis admodum conducerent, ut ineunte anno 1672 *Collinseum* literis monuit.

Hisce cogitationibus detentum mors præmatura eripuit,

ſed ejus adverſaria (ſaltem quæ in manus noſtras delapſa) præter methodi hujus exempla quædam, ipſâ tamen methodo et operandi formâ deſtituta, nihil quod ad hanc rem ſpectaret continebant. Invenienda itaque erat methodus iſtis ſeriebus producendis apta, quam etiam illuſtravimus elegantioribus aliquot exemplis figurarum quarum, apud Geometras frequentior et celebrior eſt contemplatio, quamque ingenuis candidorum Geometrarum et benevolis ſtudiis hic exponimus.

Secundum receptam dignitatum notationem, 1 eſt exponens dignitatis primæ hoc eſt ipſius radicis, 2 dignitatis ſecundæ, 3 tertiæ, 0 ipſius unitatis; ideſt exponentes ſunt ut Logarithmi, quare conſequens eſt ut $\frac{1}{2}$ ſit exponens dignitatis ſubſecundæ, $\frac{1}{3}$ ſubtertiæ, ideſt $a^{\frac{1}{2}}$ æquivalet $\sqrt{a}$, $a^{\frac{1}{3}}$ ipſi $\sqrt{c:a}$ &c: quoniam ut 1. $\sqrt{a}$, a, ſunt in ratione geometricâ, ita 0. $\frac{1}{2}$. 1. ſunt in ratione Arithmeticâ; & ut 1. $\sqrt{ca}$, $\sqrt{ca^2}a$, ſunt continuè proportionales geometricæ, ita 0, $\frac{1}{3}$, $\frac{2}{3}$, 1, ſunt continuè proportionales Arithmeticæ. Eodem modo ſi exprimenda veniat $\sqrt{a^2}$ erit $a^{\frac{2}{3}}$, & $\sqrt[3]{a^4}$ erit $a^{\frac{4}{3}}$ & ita de cæteris. Et ſicut continuâ multiplicatione per ipſam radicem poteſtates augentur, earumque exponentes a nihilo verſus ſuperiora recedunt; ita diviſione continuâ per eandem diminuuntur poteſtates earumque exponentes à nihilo verſus inferiora recedunt; hoc eſt $\frac{1}{x}$ notanda

erit

erit $X^{-1}$, $\frac{1}{X^2}$ erit $X^{-2}$, &c. Eodem $\frac{1}{\sqrt[3]{X^5}}$ erit $X^{-\frac{5}{3}}$, & $\frac{1}{\sqrt[3]{X^4}}$ $X^{-\frac{4}{3}}$, & ita de cæteris. Nam ſicut X, 1, $\frac{1}{X}$, $\frac{1}{X^2}$ ſunt in continuâ proportione Geometricâ, ita 1, 0, —1, —2 ſunt in continuâ proportione Arithmeticâ. Nos utrumque notationis modum indiſcriminatim uſurpamus.

Hoc uti decebat prænotato rem ipſam aggredimur.

Ad dimenſionem cujuſvis figuræ nihil requiritur præter inventionem ſummæ elementorum ejuſdem, per *Elementum* intellige, partem cujus acceſſione augetur figura, aucto axe per partem ſui unam indefinitè parvam. Hoc clariùs explicandum in unaquâque figuræ ſpecie.

Sit figura plana rectâ ABC et curvâ AIE comprehenſa, hujus figuræ elementum voco lineam vel rectangulum latitudinis indefinitè parvæ BCKI, quæ minimè differt a BCEI cujus acceſſione augetur figura ABI, ut fiat ACE aucto axe per partem indefinite parvam BC. Fig. 1.

Eodem modo in lineâ curvâ AIE, quam tangit FE in E, elementum voco curvulam IE, vel quæ ab illâ minimè differt particulam tangentis DE inter CE et BI productam interceptam.

Si figura prædicta vertatur circa axem AC, ut producatur ſolidum, hujus elementum voco circulum vel cilindrum rotatione ipſius BCKI productum; et ſuperficiei hujus ſolidi elementum voco ſuperficiem conicam à curvulâ IE, vel rectulâ DE prædictâ converſione deſcriptam.

Quamvis elementi quantitas in variis figuræ locis varia sit, quia tamen ejus valorem (in æquatione figuræ naturam denotante) quantitas x hujusve potestates ingrediuntur, eadem expressio in curvâ quâvis Geometricâ illi ubique insetvit; cum per x intelligatur illa respectiva axis longitudo huic respectivo elemento congruens.

Omnia hec minutim prosequi necesse non est; cùm manifesta satis sint iis qui methodo indivisibilium assueti sunt.

Reliquum est ut ex dato hoc elemento (quodcunque tandem illud sit, sive punctum, sive linea, recta aut curva, sive superficies, plana aut gibba) quod L vocabimus, axi x suo modo applicato, per x & cognitas utcunque expressio, ostendamus quomodo summa elementorum, hoc est ipsius figuræ area sit inveniendæ: in quem finem præmittendum est hoc

*Lemma.*

Quâvis rectâ in partes æquales innumeras discerptâ, summa quarumvis dignitatum, ab innumeris istis rectis ab extremitate propositæ rectæ continuò incipientibus, genitarum, equalis est rectæ propositæ potestati, quæsitis potestatibus proximè superiori, divisæ per suum exponentem.

Hoc demonstratum habetur apud Geometras. Quibus premissis, sit figura geometrica propositæ, in quâ relatio quæ est inter x & L exprimatur hâc æquatione $L = M * x^{\frac{r}{s}}$,

Si

Si jam x discerpatur in partes æquales innumeras, atque ad divisionum puncta suo modo applicentur respectiva elementa L; evidens est elementorum summam, ideſt aream quæsitam, æquari summæ dignitatum quarum exponens est $\frac{P}{R}$ ipsarum respectivè x, ductæ in datam & invariabilem M; hoc est summæ dignitatum quarum exponens $\frac{P}{R}$. partium incipientium ab initio propositæ & datæ x. ductæ in M, que ex precedente lemmate æquatur

$$M \times \frac{X^{\frac{P}{R}+1}}{\frac{P}{R}+1} = \frac{R\,M \times X^{\frac{P+R}{R}}}{P+R}$$

Unde si valor ipsius elementi ex pluribus hujusmodi terminis constet, area quæsita etiam constabit ex areis a singulis terminis prædicto modo genitis.

Hæc autem pluribus è vario figurarum genere petitis exemplis illustrare conveniet.

In Parabolâ Apollonianâ AC, cujus axis AB detur æqualis X, parameter = P, erit ordinata BC (reliquæque rectæ huic parallelæ) elementum = $P^{\frac{1}{2}} X^{\frac{1}{2}}$ unde area ABC = $\frac{2}{3} P^{\frac{1}{2}} X^{\frac{3}{2}}$. Fig. 10.

Reliquis manentibus, sit figura ABC parabola talis, ut $P^2 \times = L^3$, quare $L = P^{\frac{2}{3}} X^{\frac{1}{3}}$; unde area ABC = $\frac{3}{4} P^{\frac{2}{3}} X^{\frac{4}{3}}$.

Ad eundem modum omnes parabolas dimetiri licet, &

elicitur

elicitur Prop. 54. Geometriæ partis Univerſalis *Jac. Gregorii.*

Fig. 2. In Hyperbolâ Appollonianâ cujus centrum A, aſymptoti AB AD, detur AB = X, quæritur figura interminata ABCDD. Erit BC elementum, & $L\,X = A^2$, unde $L = A^2 X^{-1}$, eritque figura propoſita $= \frac{1}{0} A^2 X^{\frac{0}{1}} =$ infinito.

Reliquis manentibus, ſit jam $L^2 X = A^3$. Ideſt $L = A^{\frac{3}{2}} X^{\frac{-1}{2}}$, erit area interminata ABCDD $= 2 A^{\frac{3}{2}} X^{\frac{1}{2}}$. vel ſit $L X^2 = A^3$, ideſt $L = A^3 X^{-2}$ eritque area $= -A^3 X^{-1} =$ negativo. In quo caſu ſpatium ABCDD eſt ex eorum numero quæ Geometræ vocant pluſquam-infinita: ſed mutato ſigno æquatur figuræ BCEE, ex alterâ parte ipſius BC jacenti, interminatæ verſus E. Hæ duæ ſunt una & eadem hyperbola at diverſo ſitu poſita. Eodem modo reliquæ hyperbolæ tractantur, & deducitur Prop. 59. Geom. part. Univers.

Fig. 3. Sit Iterum parabola cujus vertex A, axis AB, parameter = A, AC ipſi AB perpendicularis = B. CE axi parabolæ parallela etiam data = X, EF = L. Erit $A^{\frac{1}{2}} X^{\frac{1}{2}} - B = L$. Quare area $= \frac{2}{3} A^{\frac{1}{2}} X^{\frac{3}{2}} - B X =$ ſpatio DFE — ADC.

Fig. 4. Sin ſpatium DEF ſolum quæratur, aliæ poſitiones adhibendæ, exempli gratiâ. In eâdem parabolâ cujus vertex A, axis AB, parameter = A, FB = B, DE parallela AB = L, datur

datur FE = X, erit $L = \frac{2BX}{A} - \frac{X^2}{A}$, unde area DEF = $\frac{BX^2}{A} - \frac{X^3}{3A}$.

Rurſus ſit parabola, cujus axis AB, parameter = A, AE = B, EC = X, erit $CD = L = \frac{X^2}{A} - \frac{2BX}{A} + \frac{B^2}{A}$ & area $= \frac{X^3}{3A} - \frac{BX^2}{A} + \frac{B^2X}{A}$ = figuræ AEF + ACD, ſi punctum A cadat inter E & C vel = AEF − ACD, ſi C cadat inter E & A. Fig. 5.

Hæc generatim accepta velim, in omni etenim figurâ præſentibus hiſce conditionibus obtinent: exemplis autem in Parabolâ et Hyperbolâ majoris perſpicuitatis gratiâ uſi ſumus.

Plurima alia ſymptomata generalia productam aream afficiunt pro ratione poſitionum, quæ tamen conſulto omiſimus brevitati ſtudentes.

Hoc loco viſum eſt adſcribere dimenſionem familiæ cujuſdam curvarum, inter Geometras ſatis celebris.

Sit Curva ADB ita verſus rectam AB relata, ut ductâ utcunque DC perpendiculari in AB, fiat poteſtas cujus exponens q ipſius AB, ad ſimilem poteſtatem partis AC, ut poteſtas cujus exponens p ipſius BC, ad ſimilem poteſta- Fig. 6.

tem applicatæ DC, vocetur AB A

AC x.

Et quæratur area ADC.

Erit CD L.

Ex naturâ curvæ, erit $A^q : x^q :: A-x^p : L^p$.

ergo $L = \frac{A-x \times x^p}{A^{\frac{q}{p}}}$.

Adeoque area ADC $= \frac{p \times x^{\frac{q+p}{p}}}{q+p} - \frac{p \times x^{\frac{q+2p}{p}}}{q+2p} A^{\frac{q}{p}}$.

Si quæratur area totius figuræ ADB, ponatur in præcedente æquatione A loco x, provenietque figura ADB $= \frac{A^2 p^2}{q^2+3pq+2p^2}$ Unde ratio figuræ ad quadratum axis est $p^2$ ad $q^2 + 3pq + 2p^2$.

Reliquis manentibus sit nunc $A^q - x^q \times x^p = L \times A^{q+p-1}$:

Erit $L = \frac{A^q x^p - x^{q+p}}{A^{q+p-1}}$.

Unde area figuræ ADC $= \frac{A^q x^{p+1}}{p+1 \times A^{q+p-1}} - \frac{x^{q+2p+1}}{q+p+1 \times A^{q+p-1}}$.

Methodo supra traditâ invenietur ratio quadrati axis ad figuram integram ADB, eadem quæ $q+p+1 \times p+1$ ad q.

Hæc

Hæc duo curvarum genera metitur Clariſſimus *Sluſius* in ſuis *Miſcellaneis*.

Præter has et infinitas alias figuras planas, quarum areæ per hanc methodum geometricè determinantur; dantur & plurimæ, quarum quadraturæ a *Circuli*, *Hyperbolæ* &c: quadraturis dependent, in quibus tamen ſi proportio inter aream et circulum quendam innoteſcat, ipſarum dimenſio latere non exiſtimatur. Modum igitur hujus proportionis per methodum prædictam exquirendæ infinitarum ſpiralium exemplo illuſtrabimus.

Sit *ſpiralis* C G D A talis naturæ, ut eadem ſit ratio dignitatis cujus exponens p totius circumferentiæ circuli A E H A, ad eandem arcûs A H E, quæ eſt dignitatis cujus exponens q radii C E, ad eandem partis C D. & quæratur ratio ſpatii B F D G C ad Sectorem contermin̄um C D F B. Fig. 7.

Ponatur C A = r.

& Peripheria circuli cujus radius C A = c.

C D = x.

Erit arcus D F B ſpatii propoſiti elementum.

Ex Natura Spiralis data eſt $x^{\frac{q}{p}} : r^{\frac{q}{p}} :: c : \frac{r^{\frac{q}{p}} c}{x^{\frac{q}{p}}}$ = arcui A H E.

Ergo $\frac{x^{\frac{q}{p}} C}{R^{\frac{q}{p}}}$ = arcui A H E.

Porro ut $R : x :: \frac{X^{\frac{q}{p}}}{\frac{q}{R^p}} C : \frac{C \times \frac{q+p}{X^p}}{\frac{q+p}{R^p}} =$ arcui DFB = L.

Quare figura $BFDGC = \frac{PC - X \frac{q+2p}{p}}{q+2p - R \frac{q+p}{p}}$

Ipse vero Sector Circuli $CDFB = \frac{C \times \frac{q+2p}{p}}{2R \frac{q+p}{p}}$

Quare sector Circuli est ad complementum spiralis conterminum sicut $q + 2p$ ad $2p$.

Idemque ad ipsum spatium spirale CGD, ut $2p + q$ ad $q$

Hæc est spiralis de quâ, *Prop*: 64: *Geom: Part: Univers*: eodemque modo tractari possent spirales *Prop*: 65: & 66: ibidem.

Atque ita methodo hâc exemplis circa figuras planas satis illustratâ, proximum est ut eam ad Curvarum longitudines (quamvis et has ad figuratum planarum curvis vel Curvâ & Rectis comprehensarum quadraturas reduci ex Geom: part: Univers: satis innotescat) directò investigandas, applicemus; ad altas figurarum species nempe solida horumque superficies ordine postea transituri.

Fig. 1. Sit ergo A C Q curva parabolica talis naturæ ut

posito A B x.

B C z.

a linea data.

Sit

Sit ſemper $x^3 = a z_2$. Oporteatque curvæ A C longitudinem invenire. Tangat recta D C curvam in C. facilè invenitur eſſe $DB = \frac{3}{3} x$.

$$\text{Unde } DC = \sqrt{\frac{4}{9} x^2 + \frac{x^3}{2}}.$$

Cumque elementum rectæ A B, (ideſt unitas ſeu alterum latus rectanguli latitudinis indefinitè parvæ) ſit ad elementum curvæ A C Q (ideſt pnnctum C ſicut hactenus explicatum) ſicut D B ad D C, erit Curvæ Elementum $= \sqrt{1 + \frac{9x}{4z}}$.

$$\text{Unde longitudo Curvæ } AC = \overline{a}^{\frac{1}{2}} \times \overline{\frac{4}{9} a + x}^{\frac{3}{2}} - \frac{8}{27} a.$$

Per eandem methodum invenitur longitudo curvæ A C poſito $x^5 = a z^4$, immo & ſuperiorum generum, cùm nimirum exponens ipſius x eſt numerus quivis impar, & exponens ipſius z eſt numerus proximè minor. Sicut factum eſt prop. 58. Geom. part. Univerſ.

Sit Cyclois primaria A B C cujus generator circulus A E G, & exponatur Curvæ A B longitudo inveſtiganda. Fig. 9.

Conſtat punctum D eſſe ad punctum B ut D F ad F B, id eſt ut D A ad A E, ob A E F B parallelas.

$$\text{Vocetur } GA \quad D.$$
$$AD \quad x.$$
$$\text{erit } AE = D^{\frac{1}{2}} x^{\frac{1}{2}}.$$

$$\text{Et ut } x : D^{\frac{1}{2}} x^{\frac{1}{2}} :: 1 : D^{\frac{1}{2}} x^{-\frac{1}{2}} = 2\, AE.$$

Quod ſi in hâc æquatione ponatur D loco x, fiet tota

Curva ABC = 2 D = 2 AG. Sicut demonstratur prop. 63. Geom. part. Univers.

Fig. 10. Pro dimensione superficiei, rotatione rectæ vel curvæ AC circa axem AB genitæ; adnotetur perpetim esse elementum æquale puncto C, ducto in peripheriam circuli cujus radius BC. Hoc est $\frac{\pi}{\rho}$ DC: posito DC curvæ in C normali. Per $\frac{\pi}{\rho}$ intelligitur ratio circumferentiæ circuli ad suum radium.

Exemplo sit *superficies conoïdis Parabolicæ* rotatione curvæ Parabolicæ AC, circa axem AB genitæ.

Sit parameter a.
AB data x.
erit BC = $\sqrt{ax}$.
et BD = $\frac{1}{2}$a.
Adeoque DC = $\sqrt{\frac{1}{4}a^2 + ax}$.
Quare elementum = $\frac{\pi}{\rho} \times \sqrt{\frac{1}{4}a^2 + ax}$.

Et valor *superficiei* quæsitus = $\frac{\pi}{\rho} \times \frac{2}{3}a^{\frac{1}{2}} \times \overline{x + \frac{1}{4}a}^{\frac{3}{2}}$ $\frac{\pi}{\rho} \times \frac{2}{3}a^{\frac{1}{2}} \times \frac{1}{8}a^{\frac{3}{2}}$.

Hinc fluit praxis *Prop*: 46: *Geom: Part: Univers*: Eâdem ratione dimetiri licet superficies sphæroïdis longæ & latæ, Conoïdis Hyperbolicæ, &c.

Fig. 11. Eâdem Methodo solidorum contenta exquirentur. Exemplo sit solidum à *spirali Archimedeâ* ita genitum. Sit ista spiralis

ſpiralis ex unâ revolutione orta AHB, circulus congruens BCD. Super figurâ planâ AHIB, ſpirali AHB, rectâque AB, contentâ; erigatur cylindricus rectus terminatus ad ſuperficiem coni recti, cujus baſis eſt circulus BCD, altitudo recta AE, æqualis radio AB, quæritur hujus ſolidi contentum: ideſt quam proportionem habeat ad ſphæram cujus radius eſt AB.

Vocetur AB = AE a.
Sit AO x.
erit BO = OZ = a — x.

eritque ſuperficies cylindrica ZSOI figuræ elementum, quod ut inveniatur adnotetur eſſe

Propter helicem. $\rho : \pi :: a - x : \frac{a\pi - \pi x}{\rho} = \text{arc: BC.}$

Et propter circulum. $a : x :: \frac{a\pi - \pi x}{\rho} : \frac{a\pi x - \pi x^2}{a\rho} = \text{arc: OI.}$

& arc:OI × rect:OZ = ZSOI = $L = \frac{a^2\pi x - 2a\pi x^2 + \pi x^3}{a\rho}$.

Quare ſolidum ZSOIHAE = $\frac{a\pi x^2}{2\rho} - \frac{2\pi x^3}{3\rho} + \frac{\pi x^4}{4a\rho}$.

Si in hâc æquatione loco x ponatur a, fiet ſolidum integrum = $\frac{\pi a^3}{12\rho}$: Sed ſphæra cujus radius a, = $\frac{2}{3}\frac{\pi a^3}{\rho}$. Quare ſolidum propoſitum eſt ad ſphæram ut 1 ad 8.

Eodem

Fig. 4. Eodem modo ſi parabola (cujus axis AB, ordinata ad axem BF) vertatur circa ordinatam, ex datis

Parametro = d.

BF = a.

FE = x.

Quæritur ſolidum rotatione ſpatii FED circa axem FB genitum. Erit ED $= \frac{2ax - x^2}{d}$. & circulus ſuper radio ED = L = $\frac{2a^2\pi x^2}{d^2\rho} - \frac{2a\pi x^3}{d^2\rho} + \frac{\pi x^4}{2d^2\rho}$.

Unde Solidum propoſitum $= \frac{2a^2\pi x^3}{3d^2\rho} - \frac{a\pi x^4}{2d^2\rho} + \frac{\pi x^5}{10d^2\rho}$.

Artificio jam ſæpe repetito, invenietur ſolidum ab integrâ figurâ FBA rotatione circa FB genitum $= \frac{4\pi a^5}{15d^2\rho}$.

Cylindrus autem ejuſdem baſis & altitudinis eſt $\frac{\pi a^5}{2d^2\rho}$.

quare hic ad illud ut 15 ad 8.

Hucuſque nihil occurrit quod methodo huic non ſtatim cedat, ſed cum infiniti ſint caſus, in quibus L vel x, vel utraque adeo implicetur, ut fruſtra quis in methodo illic applicandâ deſudet, niſi prius valor elementi convenienti operatione in eam formam tranſmutetur, quæ præcedentium inſtar tractari non recuſet; idcirco paucis dicendum eſt & unde anomalia iſta proveniat, & quibus modis debeat amoveri. Hæc autem difficultas oritur vel ex eo, quòd x in valore

elementi

elementi implicetur, ob non abſolutas aut diviſionem aliquam, aut radicis cujuſdam ex propoſitâ quantitate extractionem: vel quòd X & L earumve dignitates, ſecum invicem aut cum aliis, in æquatione ita involvantur, ut abſque propoſitæ æquationis reſolutione valor ipſius elementi nequeat exhiberi, reliquum igitur eſt ut exempla apponamus in quibus hæ difficultates occurrunt, unà cum methodo ipſas amoliendi, idque quantum brevis exercitationis limites concedent.

Sit *Hyperbola* KDE cujus aſymptoti AC AF, angulum rectum CAF comprehendant. Vertex ſit K, & GK BD, EC parallelæ AF, Fig. 12.

Ex datis AG = GK m.
AB a.
BC x.

Quæritur area BCED, rectis DB, BC, CE & curvâ DE comprehenſa.

Propter Hyperbolam, recta EC ſive elementum in hâc figurâ = $\frac{m^2}{a+x}$ quoniam autem in hoc elementi valore, x aliis quantitatibus ob diviſionem imperfectam ita implicatur, ut per methodum prædictam tractari nequeat; idcirco perficienda eſt diviſio ipſo opere ad modum quem docuit *Clariſſimus Mercator*; eritque

$$\frac{m^1}{a} \cdot \frac{m^2 x}{a^2} + \frac{m^3 x^2}{a^3} + \frac{m^4 x^3}{a^4} + \frac{m^5 x^4}{a^5} \text{ &c. in infinitum.}$$

Unde *area Hyperbolica* BCED

$$= \frac{m^1 x}{a} - \frac{m^2 x^2}{2a^2} + \frac{m^3 x^3}{3a^3} - \frac{m^4 x^4}{4a^4} + \frac{m^5 x^5}{5a^5} \text{ &c. in infinitum.}$$

Quòd si puncta B & G coincidant, ponaturque AG = 1, series multò simplicior evadet. Erit nimirum Area

$$BCED = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \text{ &c.}$$

Hujus quadraturæ Auctor est *Clariss: Geom: Nicolaus Mercator* in suâ *Logarithmotechniâ*.

Quo major fuerit ratio ipsius AB ad BC, eo quivis præfinitus terminorum hujus seriei numerus ad veritatem propius accedit.

At si è contra BC major fuerit quam AB, reliquis manentibus, AC dicatur a, erit $BD = L = \frac{m^1}{a - x}$ (etenim hoc casu, initium basis est punctum C). Quare Area $BCED = \frac{m^1 x}{a} + \frac{m^2 x^2}{2a^2} + \frac{m^3 x^3}{3a^3} + \frac{m^4 x^4}{4a^4}$ &c.

Utriusque hujus Methodi demonstrationem Apagogicam videre licet in *Excercit: Geom: Jac: Gregorii*.

Adnotari potest, quòd sæpissimè posità unitate loco altorius quantitatis, (ut hic factum vides) series redditur simplicior.

In

In hâc eâdem Hyperbolâ ſed diverſo modo conſideratâ invenire quoque licet & caſum quo in ipſius elementi valore, x cum aliis quantitatibus per *Aſymetriam* involvitur ut extractione radicis quadratæ ejuſque (eâdem operâ) in ſeriem infinitam tranſmutatione opus ſit.

Sit ergo *Hyperbola* D G, cujus centrum B, axis tranſverſus D E, conjugatus A C. Fig. 13.

Ex datis A B c.
D B R.
B F x.

Quæritur *area* D B F G rectis D B B F F G & curvâ Hyperbolicâ D G comprehenſa.

Erit $F G = L = \frac{R}{C} \sqrt{c^2 + x^2}$.

Quoniam autem in hoc elementi valore, x per aſymmetriam involvitur, ideo hinc liberanda, radicis quadratæ ipſo opere ex quantitate $c^2 + x^2$ extractione, cujus Operationis *Schema* hic exhibeo.

$$c^2+x^2\left(c+\tfrac{1}{2}c^{-1}x^2-\tfrac{1}{8}c^{-3}x^4+\tfrac{1}{16}c^{-5}x^6-\tfrac{5}{128}c^{-7}x^8+\tfrac{7}{256}c^{-9}x^{10}-\tfrac{21}{1024}c^{-11}x^{12}+\text{&c.}\right.$$

$$c^2$$

$$0+x^2$$

$$+x^2+\tfrac{1}{2}c^{-2}x^4$$

$$0-\tfrac{1}{4}c^{-2}x^4$$

$$-\tfrac{1}{4}c^{-2}x^4-\tfrac{1}{8}c^{-4}x^6+\tfrac{1}{64}c^{-6}x^8$$

$$0\qquad+\tfrac{1}{8}c^{-4}x^6+\tfrac{1}{64}c^{-6}x^8$$

$$+\tfrac{1}{8}c^{-4}x^6+\tfrac{1}{16}c^{-6}x^8-\tfrac{1}{64}c^{-8}x^{10}+\tfrac{1}{256}c^{-10}x^{12}$$

$$0\qquad-\tfrac{5}{64}c^{-6}x^8+\tfrac{1}{64}c^{-8}x^{10}-\tfrac{1}{256}c^{-10}x^{12}$$

$$-\tfrac{5}{64}c^{-6}x^8-\tfrac{5}{128}c^{-8}x^{10}+\tfrac{5}{512}c^{-10}x^{12}$$

$$+\tfrac{7}{128}c^{-8}x^{10}-\tfrac{7}{512}c^{-10}x^{12}$$

$$+\tfrac{7}{128}c^{-8}x^{10}+\tfrac{7}{256}c^{-10}x^{12}$$

$$-\tfrac{21}{512}c^{-10}x^{12}$$

$$0$$

Nempe

Nempe primo extrahenda radix quadrata ex primo termino $c^2$ ſc. c, & reponenda in quotiente, ac ejus quadratum ex quantitate propoſitâ ſubſtrahendum; remanet $+ x^2$. deinde quotiens c duplicandus, & reſiduum (vel primus reſidui terminus ſi ſint plures) per illum dividendum, quotuſque $+ \frac{1}{2} c^{-1} x^2$ in communi quotiente locandus. Quotiens hic noviſſimus, per præcedentis quotientis duplum & per ſeipſum multiplicandus, ac productum $+ x^2 + \frac{1}{4} c^{-2} x^4$ ex priori reſiduo $+ x^2$ ſubducendum, & hoc ultimum reſiduum $- \frac{1}{4} c^{-2} x^4$ loco prioris habendum, ideſt per totius prioris quotientis duplicati $+ 2c + c^{-1} x^2$, primum terminum 2c dividendum; & quotiens $- \frac{1}{8} c^{-3} x^4$ commum quotienti adjectus, in priorem quotum duplicatum & in ſe ductus, dat productum $- \frac{1}{4} c^{-2} x^4 - \frac{1}{8} c^{-4} x^6 + \frac{1}{64} c^{-6} x^8$, a priori reſiduo $- \frac{1}{4} c^{-2} x^4$ ſubducendum, & hoc reſiduum ſcilicet $+ \frac{1}{8} c^{-4} x^6 - \frac{1}{64} c^{-6} x^8$ ſicut priùs tractandum: idque quoties libuerit repetendum, ut in Schemate operationis videri eſt; ut & qui termini inter operandum poſſunt ommitti, cum radix quæritur cujus ultimus terminus ad determinatam quandam dimenſionem tantum aſcendat.

Eodem modo mutatis mutandis quælibet alia radix pura, ex quantitate propoſitâ extrahetur. Ex obſervatâ ſeriei cujuſvis naturâ in aliquot ab initio terminis poteſt ipſa abſque operationis tædio continuari.

Hujus radicis id est seriei, si singuli termini multiplicentur per $\frac{R}{C}$, exurget figuræ elementum

$$\frac{RC^{-2}x^2}{R} - \frac{RC^{-4}x^4}{8} + \frac{RC^{-6}x^6}{16} - \frac{5RC^{-8}x^8}{128} + \frac{7RC^{-10}x^{10}}{256} - \frac{21RC^{-12}x^{12}}{1024} + \&c.$$

Adeoque *area* DBEG =

$$\frac{RC^{-2}x^3}{6} - \frac{RC^{-4}x^5}{40} + \frac{RC^{-6}x^7}{112} - \frac{5RC^{-8}x^9}{1152} + \frac{7RC^{-10}x^{11}}{2816} - \frac{21RC^{-12}x^{13}}{53248} + \&c.$$

Fig. 13. Iisdem positis, si curva DG sit *ellipsis*, eadem series inserviet mutando tantum signa terminorum Secundi, quarti &c. In locis paribus.

Quod si ellipseos vel Hyperbolæ species determinetur, series multò simplicior evadet. Erit utique in *Circulo*, *Zona* DGNE

$$\frac{R^{-1}x^3}{2} - \frac{R^{-3}x^5}{20} + \frac{R^{-5}x^7}{56} - \frac{5R^{-7}x^9}{576} + \frac{7R^{-9}x^{11}}{1408} - \frac{21R^{-11}x^{13}}{6656} \&c.$$

Adeoque ex datâ chordâ, in Hyperbolâ ellipsi vel circulo, datur segmentum; id quod mox ex sagittâ datâ etiam assequemur.

Fig. 14. Sit ellipsis cujus axis transversus AD,

Dentur AB a.

Parameter b.

Semisegmenti ADC sagitta AC, x.

Quæritur Area ADC.

Inve-

Invenietur $DC = z = \sqrt{bx - \frac{bxx}{a}} =$

$$B^{\frac{1}{2}}X^{\frac{1}{2}} - \frac{B^{\frac{1}{2}}a^{-1}X^{\frac{3}{2}}}{2} - \frac{B^{\frac{1}{2}}a^{-2}X^{\frac{5}{2}}}{8} - \frac{B^{\frac{1}{2}}a^{-3}X^{\frac{7}{2}}}{16} - \frac{5B^{\frac{1}{2}}a^{-4}X^{\frac{9}{2}}}{128} - \text{&c.}$$

Unde *area* $ACD =$

$$\frac{2B^{\frac{1}{2}}X^{\frac{3}{2}}}{3} - \frac{B^{\frac{1}{2}}a^{-1}X^{\frac{5}{2}}}{5} - \frac{B^{\frac{1}{2}}a^{-2}X^{\frac{7}{2}}}{28} - \frac{B^{\frac{1}{2}}a^{-3}X^{\frac{9}{2}}}{72} - \frac{5B^{\frac{1}{2}}a^{-4}X^{\frac{11}{2}}}{704} - \text{&c.}$$

Reliquis manentibus, ſi figura propoſita ſit Hyperbola, eadem ſeries inſerviet, mutando tantum ſigna terminorum ſecundi, quarti &c. in locis paribus. Fig. 15.

Quòd ſi ellipſeos vel Hyperbolæ ſpecies determinetur, ſeries evadet multò ſimplicior; erit exempli gratiâ in circulo, totum ſegmentum ſc: ipſius A D C duplum Fig. 14.

$$\frac{4a^{\frac{1}{2}}x^{\frac{3}{2}}}{3} - \frac{2a^{-\frac{1}{2}}x^{\frac{5}{2}}}{5} - \frac{a^{-\frac{3}{2}}x^{\frac{7}{2}}}{14} - \frac{a^{-\frac{5}{2}}x^{\frac{9}{2}}}{36} - \frac{5a^{-\frac{7}{2}}x^{\frac{11}{2}}}{352} - \text{&c:}$$

Vel ponendo $a\,x = D^2$, & $\frac{1}{2}\,a = R$.

$$\frac{4DX}{3} - \frac{2D^{-1}X^3}{5} - \frac{R^{-1}D^{-1}X^4}{28} - \frac{R^{-2}D^{-1}X^5}{144} - \frac{5R^{-3}D^{-1}X^6}{2816} - \text{&c:}$$

Ut autem hujus Methodi univerſalitas appareat, quodque quadraturis ſectionum conicarum non coërceatur, verum ad curvas quantumvis compoſitas extendi poſſit, proponatur Hyperbola ſecundi generis nimirum.

Sit

Fig. 15. Sit Curva A D, (cujus axis A B, ordinata ad axem C D) talis naturæ; ut sit perpetuò B C · A C q. = C D cubo

Dentur A B s

A C x.

Quæritur area A C D.

Erit ex naturâ curvæ L = $\sqrt{c:sx^2+x^3}$

Extrahatur jam radix Cubica ipsâ operatione ex $sx^2+x^3$.

Invenieturque $L = s^{\frac{1}{3}}x^{\frac{2}{3}} + \frac{1}{3}s^{\frac{2}{3}}x^{\frac{5}{3}} - \frac{1}{9}s^{\frac{5}{3}}x^{\frac{8}{3}} + \frac{5}{81}s^{\frac{8}{3}}x^{\frac{11}{3}}$ &c.

Adeoque area quæsita A C D, erit

$\frac{3}{5}s^{\frac{1}{3}}x^{\frac{5}{3}} + \frac{1}{8}s^{\frac{2}{3}}x^{\frac{8}{3}} - \frac{1}{33}s^{\frac{5}{3}}x^{\frac{11}{3}} + \frac{5}{171}s^{\frac{8}{3}}x^{\frac{14}{3}}$ &c.

Quoniam autem *Conchoidum* Contemplatio, ut pervetusta, ita etiam inter Geometras cum primis celebris extitit, idcirco illa hic sibi locum meritò vendicat.

Fig. 16. Sit *Conchois Nicomedis prima* A E, polo C, vertice A, normâ B H descripta; sintque polus & vertex æqualiter à normâ distantes. Sit K E ipsi A B, & E D ipsi B H perpendicularis.

Dentur A B = B C a.

A K x.

Quæritur area A K E.

Liquet esse in hâc figurâ F E q = A B q = D E q + D F q, idest quadratis ipsius A B — A K, & ipsius B D — B F hoc est.

$a^2 =$

$$a^3 = a^3 - 2ax + x^2 + L^2 - \frac{2aL^2}{2a-x} + \frac{a^2L^2}{4a^2 - 4ax + x^2}$$

quæ æquatio ſi ordinetur & reducatur, erit

$$L = \frac{2a-x}{a-x} \times \sqrt{2ax - xx}.$$

Et poſt reductionem, poſitâque $D = 2a$, erit

$$L = 2D^{\frac{1}{2}}x^{\frac{1}{2}} + D^{-\frac{1}{2}}x^{\frac{3}{2}} + \frac{11}{8}D^{-\frac{1}{2}}a^{-1}x^{\frac{5}{2}} + \frac{45}{32}D^{-\frac{1}{2}}a^{-2}x^{\frac{7}{2}}, \&c.$$

Unde area AKE =

$$\frac{4}{3}D^{\frac{1}{2}}x^{\frac{3}{2}} + \frac{2}{5}D^{-\frac{1}{2}}x^{\frac{5}{2}} + \frac{11}{28}D^{-\frac{1}{2}}a^{-1}x^{\frac{7}{2}} + \frac{45}{144}D^{-\frac{1}{2}}a^{-2}x^{\frac{9}{2}} \&c.$$

Atque ſi alio modo inveſtigetur elementum in hâc figurâ ſcil: ex additione ſinus recti & tangentis anguli, cujus ſinus verſus eſt x & radius a, quæ eſt vulgata hujus conchoidis proprietas, ideſt ex additione ipſius $\sqrt{2ax - x^2}$, et ipſius $\frac{a}{a-x} \times \sqrt{2ax - xx}$. fit elementum ut prius $\frac{2a-x}{a-x} \times \sqrt{2ax - xx}$.

Si libet ſeriem provenientem ex ductu ipſius BK in KE, ſive A − X in valorem elementi addere priori ſeriei, exſurget ſpatium AEDB.

Hoc eodem modo, vel ex analogiâ, invenietur dimenſio Conchoidis cùm vertex et polus non diſtant æqualiter à normâ.

Si cui libuerit quadraturam Conchoidis hujus primæ à *Jac. Gregoria* in *exercitationibus* ſuis *Geometricis* demonſtratam ritè proſequi, in eandem cum priori ſeriem incidet.

Fig. 16. Sit nunc *Conchois* AE *talis naturæ* ut (reliquis manentibus quæ ſupra) ſit rectangulum CF × FE = CB × BA rectangulo vel BAq. quæraturque area AKE.

Erit ex Curvæ proprietate AK : KB :: KEq : KCq. undo erit $KEq = \frac{KCq \times AK}{KB}$. hoc eſt in terminis analyticis $L^2 = \frac{4a^2x - 4ax^2 + x^3}{a-x}$; ergo

$$L = \frac{\sqrt{4a^2x - 4ax^2 + x^3}}{\sqrt{a-x}}$$

In quo caſu & ſimilibus, ad operationis facilitatem conducit utramque partem per denominatorem multiplicare, eritque hoc facto

$$\frac{\sqrt{4a^3x - 8a^2x^2 + 5ax^3 - x^4}}{a-x} = \frac{2a-x}{a-x} \times \sqrt{ax - x^2} =$$

$$2a^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{1}{4}a^{\frac{1}{2}}x^{\frac{3}{2}} + \frac{1}{4}a^{\frac{3}{2}}x^{\frac{5}{2}} + \frac{7}{64}a^{\frac{5}{2}}x^{\frac{7}{2}} + \frac{7}{64}a^{\frac{7}{2}}x^{\frac{9}{2}} \text{ &c.}$$

Quare area AKE = $\frac{4}{3}a$

$\frac{4}{3} a^{\frac{1}{2}} x^{\frac{3}{2}} + \frac{1}{24} a^{-\frac{3}{2}} x^{\frac{7}{2}} + \frac{3}{128} a^{-\frac{5}{2}} x^{\frac{9}{2}} + \frac{15}{352} a^{-\frac{7}{2}} x^{\frac{11}{2}}$ &c.

Ex comparatione elementorum harum Conchoidum facilè patet quòd ſicut in Conchoide Nicomedis primâ, ſolidum ex revolutione interminati ſpatii Semiconchoidalis circa Normam, æquatur ſolido genito ex rotatione quadrantis circuli centro B radio B A deſcripti, circa parallelam normæ per polum C tranſeuntem, ita ſolidum ortum ex revolutione interminati ſpatii Semiconchoidalis hujus Secundi circa ipſius Normam, æquatur Solido orto ex revolutione ſemicirculi Circa diametrum A B deſcripti circa parallelam normæ per polum C itidem tranſeuntem; uti demonſtravit *Clariſs. Sluſius* in ſuis *Miſcellaneis*.

Sit jam *Caſſois* K M ſemicirculo K D G congruens, Fig. 17.

Dentur K G a.

K I x.

Quæritur Area K I L.

Erit I L normalis in K G = L.

Eſt ex hujus Curvæ Naturâ I D : I K : : I K : I L, hoc eſt in terminis analyticis,

$\sqrt{ax - xx} :: x :: x : L.$

Unde $L = \frac{xx}{\sqrt{ax - xx}}$.

Et reducendo, erit $L =$

$a^{-\frac{1}{2}}x^{\frac{5}{2}} + \frac{1}{2}a^{-\frac{3}{2}}x^{\frac{5}{2}} + \frac{3}{8}a^{-\frac{5}{2}}x^{\frac{7}{2}} + \frac{5}{16}a^{-\frac{7}{2}}x^{\frac{9}{2}} + \frac{35}{128}a^{-\frac{9}{2}}x^{\frac{11}{2}}$ &c.

Unde Area KIL =

$\frac{2}{3}a^{-\frac{1}{2}}x^{\frac{5}{2}} + \frac{1}{7}a^{-\frac{3}{2}}x^{\frac{7}{2}} + \frac{1}{18}a^{-\frac{5}{2}}x^{\frac{9}{2}} + \frac{5}{88}a^{-\frac{7}{2}}x^{\frac{11}{2}} + \frac{35}{832}a^{-\frac{9}{2}}x^{\frac{13}{2}}$ &c.

Hiſce congruit *Ciſſoidis* dimenſio allata in *excercitationibus Geometricis Jac: Gregorii*, ſi nempe areæ jam inventæ addatur triangulum ILG vel KID hoc eſt

$\frac{1}{2}a^{\frac{3}{2}}x^{\frac{3}{2}} - \frac{1}{4}a^{-\frac{1}{2}}x^{\frac{5}{2}} - \frac{1}{16}a^{-\frac{3}{2}}x^{\frac{7}{2}} - \frac{1}{32}a^{-\frac{5}{2}}x^{\frac{9}{2}} - \frac{5}{256}a^{-\frac{7}{2}}x^{\frac{11}{2}}$ &c.

& aggregatum *viz.* figura KLG,

$\frac{1}{2}a^{\frac{3}{2}}x^{\frac{3}{2}} + \frac{1}{20}a^{-\frac{1}{2}}x^{\frac{5}{2}} + \frac{1}{112}a^{-\frac{3}{2}}x^{\frac{7}{2}} + \frac{20}{384}a^{-\frac{5}{2}}x^{\frac{9}{2}} + \frac{840}{22528}a^{-\frac{7}{2}}x^{\frac{11}{2}}$ &c.

comparetur cum ſegmento circulari KND, ideſt ſemiſegmento KNDI — triangulo KDI; hoc eſt

$\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{1}{5}a^{-\frac{1}{2}}x^{\frac{5}{2}} - \frac{1}{28}a^{-\frac{3}{2}}x^{\frac{7}{2}} - \frac{1}{72}a^{-\frac{5}{2}}x^{\frac{9}{2}} - \frac{5}{704}a^{-\frac{7}{2}}x^{\frac{11}{2}}$ &c.

dempto,

$\frac{1}{2}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{1}{4}a^{-\frac{1}{2}}x^{\frac{5}{2}} - \frac{1}{16}a^{-\frac{3}{2}}x^{\frac{7}{2}} - \frac{1}{32}a^{-\frac{5}{2}}x^{\frac{9}{2}} - \frac{5}{256}a^{-\frac{7}{2}}x^{\frac{11}{2}}$ &c.

ideſt

$\frac{1}{6}a^{\frac{1}{2}}x^{\frac{3}{2}} + \frac{1}{20}a^{-\frac{1}{2}}x^{\frac{5}{2}} + \frac{3}{112}a^{-\frac{3}{2}}x^{\frac{7}{2}} + \frac{20}{1152}a^{-\frac{5}{2}}x^{\frac{9}{2}} + \frac{840}{67584}a^{-\frac{7}{2}}x^{\frac{11}{2}}$ &c.

Deprehendetur hæc ſeries præcedentis ſubtripla (ideſt terminus termini ſubtriplus) hoc eſt figura KND ſubtripla figuræ KGL.

Atque

Atque ita in planis figuris frequentiori contemplationi obverſis, demonſtratâ ratione dimetiendi ipſarum areas per ſeries infinitas, idem in longitudine curvarum quarumvis inveſtigandâ exequamur.

Sit ellipſis cujus axis AB a, parameter B, ex datâ AC x, quæritur longitudo curvæ AD. Ex *elementis*, eſt DC ordinata ad axem æqualis Fig. 14.

$$\frac{\sqrt{bx - bxx}}{a}.$$

& $CE = \frac{1}{2}b - \frac{bx}{a}$, poſitâ DE curvæ in D normali.

Eritque $DE = \sqrt{\frac{1}{4}b^2 - \frac{b^2x}{a} + \frac{b^2x^2}{a^2} + bx - \frac{bx^2}{a}}$.

Sed Curvæ propoſitæ elementum æquatur quartæ proportionali ipſis DC DE & elemento rectæ AC ſeu unitati.

Hoc eſt $L = \frac{\sqrt{\frac{1}{4}b^2 - \frac{b^2x}{a} + \frac{b^2x^2}{a^2} + bx - \frac{bx^2}{a}}}{\sqrt{bx - \frac{bx^2}{a}}}$.

Et reducendo,

Erit $L = \frac{1}{2}B^{\frac{1}{2}}x^{-\frac{1}{2}} + B^{-\frac{1}{2}}x^{\frac{1}{2}} - \frac{3}{4}B^{\frac{1}{2}}a^{-1}x^{\frac{1}{2}}$,

$- B^{-\frac{3}{2}}x^{\frac{3}{2}} + \frac{3}{2}B^{-\frac{1}{2}}a^{-1}x^{\frac{3}{2}} - \frac{5}{16}B^{\frac{1}{2}}a^{-2}x^{\frac{3}{2}}$

$+ 2B^{-\frac{5}{2}}x^{\frac{5}{2}} - \frac{9}{2}B^{-\frac{3}{2}}a^{-1}x^{\frac{5}{2}} + \frac{21}{8}B^{-\frac{1}{2}}a^{-2}x^{\frac{5}{2}} - \frac{7}{32}B^{\frac{1}{2}}a^{-3}x^{\frac{5}{2}}$ &c.

Unde (reductis ad eandem denominationem iis terminis in quibus x eſt ejuſdem poteſtatis)

Erit arcus ellipticus AD =

$$\frac{B^{\frac{1}{2}} x^{\frac{1}{2}} + 4aB^{-\frac{1}{2}} x^{\frac{3}{2}} - 3B^{\frac{1}{2}} x^{\frac{3}{2}} - 16a^2 B^{-\frac{3}{2}} x^{\frac{5}{2}} + 24a^2 B^{-\frac{1}{2}} x^{\frac{5}{2}} - 5aB^{\frac{1}{2}} x^{\frac{5}{2}}}{6aB^{\frac{1}{2}}}$$

$$\frac{+ 64a^6 B^{-2} x^{\frac{7}{2}} - 144a^4 B^{-2} x^{\frac{7}{2}} + 92a^4 B^{-1} x^{\frac{7}{2}} - 7a^3 B^{-2} x^{\frac{7}{2}}}{40a^3 B^2} \quad \&c.$$

$$\frac{}{112\, a^6 B^{\frac{9}{2}}}$$

Fig. 15. Reliquis Manentibus. ſi curva prædicta eſſet *hyperbola*, ſeries eadem illi inſerviret; ſi affirmentur omnes omnium terminorum partes, & negentur totus terminus tertius, quintus, &c. in locis imparibus.

Quod ſi Hyperbolæ vel ellipſeos *ſpecies* determinetur, ſeries evadet ſimplicior, erit *ex. gr.* in *Circulo* cujus diameter a, *arcus* cujus ſagitta x =

$$2a^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{1}{3} a^{-\frac{1}{2}} x^{\frac{3}{2}} + \frac{3}{20} a^{-\frac{3}{2}} x^{\frac{5}{2}} + \frac{5}{56} a^{-\frac{5}{2}} x^{\frac{7}{2}} + \&c.$$

Fig. 18. Sit parabola AC cujus vertex A, axis AP, AB tangens in vertice, BC normalis in AB.

Ex datis parametro P.

AB x.

Quæritur longitudo Curvæ AC.

Tangat recta CQ curvam in C,

erit

Erit $CB = \frac{x^2}{p}$.

$BQ = \frac{1}{2} x$.

Unde $QC = \sqrt{\frac{x^2 + x^4}{p^2}}$,

Eſt autem Curvæ propoſitæ elementum (reſpectu baſis AB) quarta proportionalis ipſis QB, QC, & unitati, ergo

$$1 = \frac{\sqrt{\frac{1}{4}x^2 + \frac{x^4}{p^2}}}{\frac{1}{2}x}$$

Quæ reducta ſecundum artem, dat

$L = 1 + 2p^{-2}x^2 - 2p^{-4}x^4 + 4p^{-6}x^6 - 10p^{-8}x^8$ &c.

Unde Curva $AC = x + \frac{2}{3}p^{-2}x^3 - \frac{2}{5}p^{-4}x^5 + \frac{4}{7}p^{-6}x^7 - \frac{10}{9}p^{-8}x^9$ &c.

Eadem ſeries proveniet, ſi ſecundum praxin *Prop.* 51. *Part. Geom. Univerſ.* fiat Hyperbola æquilatera, cujus centrum A, ſemiaxis AG æqualis ſemiparametro parabolæ & producatur BC, ut ipſi GK (parallelæ AB) occurrat in K, & Hyperbolæ in N, & ipſis AGKB rectangulo, AGNB ſpatio hyperbolico, AB lineæ rectæ, ideſt,

$\frac{1}{2}px; \frac{1}{2}px + \frac{1}{5}p^{-1}x^3 - \frac{1}{5}p^{-3}x^5 + \frac{2}{7}p^{-5}x^7 - \frac{1}{3}p^{-7}x^9$ &c; & x, fiat quarta proportionalis.

$x + \frac{2}{3}p^{-2}x^3 - \frac{2}{5}p^{-4}x^5 + \frac{4}{7}p^{-6}x^7 - \frac{10}{9}p^{-8}x^9$ &c.

Pro

Pro longitudine Curvæ parabolicæ quæsitâ.

Fig. 13. Exponatur jam ellipsis ADCE, cujus semiaxis DB R, BC C, Detur QG perpendicularis in axem DB vel illi æqualis BF X, Quæritur *arcus ellipticus* DG.

Ex Elementis invenietur $GF = \frac{\sqrt{R^2C^2 - R^2X^2}}{C}$.

Et (posità GS curvæ in G normali) $FS = \frac{R^2X}{C^2}$.

Unde erit $GS = \frac{\sqrt{C^4R^2 - C^2R^2X^2 + R^4X^2}}{C^2}$.

Cumque sit arcus propositi elementum, ipsis GF GS & unitati quarta proportionalis, erit

$$L = \frac{\sqrt{C^4 - C^2X^2 + R^2X^2}}{\sqrt{C^4 - C^2X^2}}.$$

Quare post reductionem, erit $L =$

$$1 + \frac{R^2X^2}{2C^4} + \frac{R^2X^4}{2C^6} - \frac{R^4X^4}{8C^8} + \frac{R^2X^6}{2C^8} - \frac{R^4X^6}{4C^{10}} + \frac{R^6X^6}{16C^{12}}$$

$$- \frac{R^2X^8}{2C^{10}} - \frac{3R^4X^8}{8C^{12}} + \frac{3R^6X^8}{16C^{14}} - \frac{5R^8X^8}{128C^{16}} \quad \&c.$$

Unde area propositæ sive *arcus ellipticus* DG erit (reductis ad eandem denominationem iis terminis in quibus x est ejusdem dignitatis)

$$x + \frac{R^2x^3}{6c^4} + \frac{4R^3c^2x^3 - R^4x^5 + 8c^4R^2x^7 - 4c^2R^4x^9 + 4R^6x^{11}}{40c^8} + \frac{}{112c^{12}}$$

$$+ \frac{64c^6R^2x^9 - 48c^4R^4x^9 + 24c^2R^6x^9 - 5R^8x^9}{1152c^{16}} \&c.$$

Reliquis Manentibus, ſi Curva prædicta eſſet Hyperbola, prædicta quoque ſeries illi inſerviret, ſi omnium terminorum partes affirmentur, & negentur totus terminus tertius, totus quintus, ſeptimus &c. in locis imparibus.

Si determinetur hyperbolæ vel ellipſeos ſpecies, ſeries hæc ſimplicior evadet, ut ſi in ellipſi $C = 2R$, erit Curva $DG =$

$$x + \frac{x^3}{96R^2} + \frac{3x^5}{2048R^4} + \frac{113x^7}{458752R^6} + \frac{3429x^9}{7549747R^8} \&c.$$

Vel in Circulo cujus radius R, Arcus cujus ſinus rectus x, erit

$$x + \tfrac{1}{6}R^{-2}x^3 + \tfrac{3}{40}R^{-4}x^5 + \tfrac{5}{112}R^{-6}x^7 + \tfrac{35}{1152}R^{-8}x^9 \&c.$$

Ex Curvæ ellipticæ vel hyperbolicæ elemento, cognoſci poteſt, ejuſdem rectificationem non dependere a quadraturâ ſpatii, ſub eâdem & rectis comprehenſi, ſicut in Circuli Curvâ obtinet. Immo ex elemento Cujuſlibet figuræ citò reducto, facilè cuivis inſpicienti patebit gradus Curvæ, cujus quadraturam ſupponit; at figuram de-

terminare, ejuſque partem ritè conſtruere, aliquâ ſæpe attentione indiget.

Libet hic adjungere ad *arcum Circularem approximationem linearem* quâ vel faciliorem, vel ſimpliciorem vix ſperem, utpote quæ ab arcûs biſectione ſolum dependeat.

Fig. 19. Sit arcus quilibet Semicirculo minor HKL, cujus chorda HL, ducatur recta HA tangens arcum in puncto H, ſitque angulus HLA rectus. Et recta HG dividat arcum HL bifariam in K, rectæ LA occurens in G, ſitque ang: HGB rectus. Et recta HF dividat arcum HK bifariam in S, rectæ GB occurrens in F, ſitque ang: HFC rectus, atque ita deinceps in infinitum.

Dico quòd Arcus HKL erit major quam HL, & minor quam HA; item major, quam HG, & minor quam HB; & major quam HF, & minor quam HC. &c

Ducatur QR recta, circulum tangens in L, & lineæ HA occurrens in R. Per K ducatur KY parallela GB, item KT perpendicularis ipſi LH, quæ ad centrum N pertinget, & rectam HL bifariam dividet in T.

Primò Manifeſtum eſt arcum LKH majorem eſſe recta LH, erit etiam minor quam HA, quoniam ang: ALR = (ang: QLX = ang: LHX = ob ſimilia triangula AHX LHX) ang: LAR erit RA = RL, quare arcus LKH minor (LR + HR =) HA.

Rurſus

Rursus quoniam KT, parallela LG, & HT = TL, erit HK = KG, sed HK recta minor est arcu HSK, quare prioris duplum *scilz.* recta HG, posterioris duplo *scilz* arcu HKL minus est.

Porro ob HK = KG, erit HY = BY, at HY major arcu HSK, (quod eodem modo ostenditur quo ostensa est HA major arcu HKL) quare tota recta HB, toto arcu HKL major est. Atque ita continuabitur demonstratio quousque libuerit.

Cum *Cycloidis* applicatæ à figurâ generatricis arcubus dependeant, ejus dimensio ante curvarum rectificationem non erat tradenda; Methodum vero procedendi (illâ traditâ) in hoc curvarum genere, exemplo *Cycloidis primariæ* ex circulo genitæ declarabimus.

Esto *Cyclois primaria* ABC cujus circulus generator AEG diametro AG descriptus, ducatur recta BD ipsi AG perpendicularis, cycloidi in B, circulo verò in E occurrens. Fig. 9.

Detur AG a

AD x

Quæritur Area figuræ ABD.

Quoniam in hâc figurâ (hoc modo consideratâ) elementum id est recta BD = DE + EB = DE + arc: AE.

Et recta $DE = a^{\frac{1}{2}} x^{\frac{1}{2}} - \frac{1}{2} a^{-\frac{1}{2}} x^{\frac{3}{2}} - \frac{1}{4} a^{-\frac{3}{2}} x^{\frac{5}{2}} - \frac{1}{16} a^{-\frac{5}{2}} x^{\frac{7}{2}}$ &c.

Item Arcus $AE = a^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{1}{6} a^{-\frac{1}{2}} x^{\frac{3}{2}} + \frac{3}{40} a^{-\frac{3}{2}} x^{\frac{5}{2}} + \frac{5}{112} a^{-\frac{5}{2}} x^{\frac{7}{2}}$ &c.

Quarũ sũma $= 2a^{\frac{1}{2}} x^{\frac{1}{2}} - \frac{1}{3} a^{-\frac{1}{2}} x^{\frac{3}{2}} - \frac{1}{20} a^{-\frac{3}{2}} x^{\frac{5}{2}} - \frac{1}{56} a^{-\frac{5}{2}} x^{\frac{7}{2}}$ &c.

Unde area $ABD = \frac{4}{3} a^{\frac{1}{2}} x^{\frac{3}{2}} - \frac{2}{15} a^{-\frac{1}{2}} x^{\frac{5}{2}} - \frac{1}{70} a^{-\frac{3}{2}} x^{\frac{7}{2}} - \frac{1}{252} a^{-\frac{5}{2}} x^{\frac{9}{2}}$ &c.

Cum hucusque supposuerimus, figuræ elementum, ad axem vel basem normaliter semper, suo modo applicari, nunc quid fieri debeat, cum angulus applicationis est obliquus, in problemate, alias inter Geometrica prænobili ostendemus.

Nam quia series præcedentes longitudini curvæ hyperbolicæ nisi prope verticem non satis inserviunt (etenim vel una vel altera ex Methodis præcedentibus cuivis ellipseos particulæ rectificandæ satis idonea est) ecce aliam quæ priorum defectus supplet.

Fig. 20. Sit Hyperbola GEF, cujus asymptoti AI AB ducantur asymptoto AI, parallelæ rectæ DG, CE BF quarum DG ipsi AD æqualis, reliquæ vero CE, BF ad libitum, fiatque GH asymptoto AB perpendicularis.

Dentur $AD = DG$ R.

$GH$ C.

$AC$ G.

$AB$ F.

Sitque $\sqrt{4R^2 - 4C^2}$ B.

Si

Si angulus IAB fuerit acutus, erit *Curva* EF.

$$= \frac{1}{F} - \frac{1}{G} + \frac{BR}{2F} - \frac{I\,BR}{2G} + \frac{4R^4 - B^2R^2}{24G^3} + \frac{B^2R^2 - 4R^4}{24F^3}$$

$$+ \frac{4BR^5 - B^3R^3}{80G^5} + \frac{B^3R^3 - 4BR^5}{80F^5}$$

$$+ \frac{24B^2R^6 + 16R^6 - 5B^4R^4}{896\,G^7} + \frac{+16R^6 + 5B^4R^4 - 24B^2R^6}{896\,F^7}$$

$$+ \frac{40B^3R^7 - 48BR^9 - 7B^5R^5}{2304\,G^9} + \frac{+48BR^9 + 7B^5R^5 - 40B^3R^7}{2304\,F^9}$$

$$+ \frac{64R^{12} - 240B^2R^{10} + 140B^4R^8 - 21B^6R^6}{11264\,G^{11}}$$

$$+ \frac{240B^2R^{10} - 64R^{12} + 21B^6R^6 - 140B^4R^8}{11264\,F^{11}} \quad \&c.$$

In hoc & ſimilibus caſibus, ſæpe præſtat aream propoſitam conſiderare, tanquam differentiam duarum aliarum, ſicut hic factum eſt.

Si vero reliquis manentibus, IAB angulus, fuerit obtuſus, prædicta ſeries ei quoque inſerviret, mutando tantum ſigna terminorum ſecundi, quarti, &c, in locis paribus. Per terminum hic intellige omnes quantitates ubi F & G ſunt ejuſdem dignitatis.

Si denique angulus IAB fuerit rectus, erit B = 0, & proinde evaneſcent omnes quantitates, in quibus reperitur B, unde ſeries evadet ſimplicior, erit enim Curva EF =

$$\frac{1}{V} =$$

$$F - G + \frac{R^4}{6G^3} - \frac{R^4}{6F^3} - \frac{R^8}{56G^7} + \frac{R^8}{56F^7} + \frac{R^{12}}{176G^{11}} - \frac{R^{12}}{176F^{11}} \text{ &c.}$$

Multa essent circa hanc & similes operationes observanda, *sc:* de areâ per solas dignitates ipsius F & G productâ, hoc est, unâ ex areis quarum area proposita est differentia; at Analysta expertus harum arearum particularia symptomata per se facilè dignoscet.

Nec opinor obscurum erit quomodo terminis anguli quantitatem denotantibus utendum sit, cum hæ necessariæ sint ad longitudinem tangentis vel ipsi normalis inter punctum contactus & basem interceptæ determinandam.

Aliæ ex eodem elemento series possunt Colligi.

Addemus & præcedentibus Consimilem *Curvæ Logarithmicæ* dimensionem.

Fig. 22. Sit *Curva Logarithmica* ML, cujus asymptotos recta KQ, ad quam ex punctis L & M, fiant perpendiculares MH LK, ducatur recta MA parallela HK.

Ex datis KL B.

HM = KA C.

HK = MA M.

Quæritur Longitudo Curvæ LM.

Nota est hujus Curvæ proprietas, quod si ex quolibet puncto L, ducatur IL tangens Curvam in L, est semper KI æqualis eædem determinatæ rectæ putà R. Unde si ipsa B ponatur basis, erit figuræ elementum

= √

$$= \sqrt{1 + \frac{R^2}{B^2}} =$$

$$1 + \frac{1}{2} B^{-2} R^2 - \frac{1}{8} B^{-4} R^4 + \frac{1}{16} B^{-6} R^6 - \frac{5}{128} B^{-8} R^8 \text{ &c.}$$

Unde area per hoc elementum deſcripta (ideſt ex ſummâ horum elementorum conſtans)

$$= B - \frac{1}{2} R^2 B^{-1} + \frac{1}{24} R^4 B^{-3} - \frac{1}{80} R^6 B^{-5} + \frac{5}{896} R^8 B^{-7} \text{ &c.}$$

Simileque productum per c æquabitur longitudini hujus Curvæ poſitâ c baſe.

Unde *longitudo Curvæ* ML (baſem habentis AL = B–c) æquabitur

$$B - C - \frac{1}{2} B^{-1} R^2 + \frac{1}{2} C^{-1} R^2 + \frac{1}{24} B^{-3} R^4 - \frac{1}{24} C^{-3} R^4$$

$$- \frac{1}{80} B^{-5} R^6 + \frac{1}{80} C^{-5} R^6 + \frac{5}{896} B^{-7} R^8 - \frac{5}{896} C^{-7} R^8 \text{ &c.}$$

Operando nimirum eo modo, quo in longitudine Curvæ Hyperbolicæ inter aſymptotos exquirendâ, factum eſt.

Sin aliter reducatur hujus curvæ elementum, ideſt ſi in extrahendâ radice, ponatur $\frac{R^2}{B^2}$ primò; proveniet elementum baſi B conveniens

$$\frac{R}{B} + \frac{B}{2R} - \frac{B^3}{8R^3} + \frac{B^5}{16R^5} - \frac{5B^7}{128R^7} \text{ &c.}$$

Et longitudo hoc elemento deſcripta

=

$$= \frac{R\,B^0}{0} + \frac{B^2}{4R} - \frac{B^5}{32R^3} + \frac{B^6}{96R^5} - \frac{5B^8}{1024R^7} \&c.$$

Eodem modo longitudo basi c conveniens erit

$$= \frac{R\,C^0}{0} + \frac{C^2}{4R} - \frac{C^4}{32R^3} + \frac{C^6}{96R^5} - \frac{5C^8}{1024R^7} \&c.$$

Quare Curva ML basi $B-C$ conveniens $=$

$$\frac{R\,B^0}{0} - \frac{R\,C^0}{0} + \frac{B^2-C^2}{4R} - \frac{C^4-B^4}{32R^3} + \frac{B^6-C^6}{96R^5} + \frac{5C^8-5B^8}{1024R^7} \&c.$$

Aſt primus terminus ſc. $\frac{R}{0}B^0 - \frac{R}{0}C^0$, æquatur ſpatio hyperbolico, comprehenſo unâ aſymptoto, curvâ hyperbolicâ, & duabus rectis, uni aſymptoto parallelis alteri vero perpendicularibus; quarum una $= \frac{R}{B}$, altera $\frac{R}{C}$: hujus hyperbolæ ſemiaxis $= \sqrt{2R}$. hoc autem ſpatium, ex natura curvæ Logarithmicæ, æquale eſt rectangulo ex M in I, ideſt M. Unde erit Curva ML

$$M + \frac{B-C}{4R} + \frac{C^4-B^4}{32R^3} + \frac{B^6-C^6}{96R^5} + \frac{5C^8-5B^8}{1024R^7} \&c.$$

Una harum ſerierum, alterius defectum ſemper ſupplet; ita ut vel una vel altera longitudini Curvæ ubivis ſatis appropinquet.

Sed quoniam inquiſitio Curvæ ex datâ tangente, magni ſubinde eſt uſus, illam *curvæ circularis* exemplo, obiter illuſtrabimus

Sit

Sit Circulus A F centro D, radio D A deſcriptus, quem tangit recta A C in A, ducatur Secans D C circulo in F, & tangenti in C occurrens. Fig 21

Ex datis D A R.
A C X.

Quæritur *arcus* A F.

Ponatur tangens A C baſis; hoc eſt dividi intelligatur in partes innumeras æquales, & à centro D ad diviſionum puncta rectæ ducantur, Curvæ circularis vel tangentis ritè ſumptæ particulæ, inter duas quaſvis proximas ex hiſce rectis interceptæ, erunt curvæ propoſitæ elementa, ſuis reſpectivè baſeos punctis applicata, *Ex: Gr:* Sit C B una ex infinitè exiguis baſeos A C partibus. Ducatur recta D B, item alia F E circulum contingens in F, & rectis D B A C occurrens punctis H & E; erit Curvula (vel rectula) H F inter rectas D B D C intercepta, reſpectivum Curvæ Circularis elementum baſeos puncto B C applicatum. Fiat H K recta parallela B C. Jam ex ſimilitudine triangulorum D A C F E C F H K, item triangulorum D H K D B C, invenietur in terminis analyticis H F ſeu

$$L = \frac{R^2 \times \sqrt{R^2 + X^2} - R^3}{\overline{R^2 + X^2}^{\frac{3}{2}} - R^3 - R X^2}$$

Et poſt reductionem neceſſariam, erit

$$L = 1 - R^{-2}X^2 + R^{-4}X^4 - R^{-6}X^6 + R^{-8}X^8 \text{ &c.}$$



Unde

Unde *arcus quæsitus* A F

$= x - \frac{1}{3} R^{-3} x^3 + \frac{1}{5} R^{-4} x^5 - \frac{1}{7} R^{-6} x^7 + \frac{1}{9} R^{-8} x^9$ &c.

Ad hunc modum Series, Sinubus, tangentibus, & Secantibus artificialibus accommodatæ, exhiberi possunt; adhibitis quæ ad hanc rem illustrandam traduntur in *Exerc: Geom: Jac. Gregorii.*

Fortassis notatu dignum erit, hinc consequi *præstantissimi Geometræ Gothofredi Gilberti Leebnitii, Circuli Quadraturam transact: Phil: Mensis Aprilis an:* 1682 editam, si nempe ponatur D A $= \frac{1}{2}$, arcus A F 45 *gr:* erit A C etiam $\frac{1}{2}$. Unde erit Sector D A F $=$ $\frac{1}{2}$ D A $\times$ A F *arc:* $=$

$$\frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{5} \times \frac{1}{2} \times \frac{1}{2} - \frac{1}{7} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{9} \times \frac{1}{2} \times \frac{1}{2} - \text{ &c.}$$

Ejusque octuplum, nimirum *Circulus cujus quadratum circumscriptum est* 1, erit

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{ &c. in infinitum.}$$

Generaliora forsitan possent elici, at hæc lubens prætereo.

Formâ applicationis hujus methodi, ad figuras planas & lineas Curvas utcunque absolutâ, restaret ut de Solidorum superficiebus, ipsisque Solidis aliquid adjiceretur; verum cum in illis nulla post prædicta supereſſe possit difficultas, unico exemplo, *Spheroidis segmenti secundi*, rem conficiam.

Sit

Sit Ellipſis, cujus axis tranſverſus A C, conjugatus D E, centrum B, Supponatur Spherois generari ex revolutione ſemiellipſeos A D C circa axem A C, ſeceturque Spherois à duobus planis parallelis per puncta K & B, axi D B normalibus, item à duobus planis parallelis per puncta B & F axi B C normalibus. Fig. 13.

Sit B F a.
B K b.
A B c.
D B r.

Si ponatur B F axis, erunt hujus Solidi elementa, *Circulorum Zonæ*, quorum radii ſunt reſpectivè applicatæ F G latitudo autem communis B K = b, patet vero eſſe

$$GF = \frac{\sqrt{r^2c^2 - r^4a^2}}{c}$$

$$= r - \frac{ra^2}{2c^2} - \frac{ra^4}{8c^4} - \frac{ra^6}{16c^6} - \frac{5ra^8}{128c^8} \text{ &c.}$$

Conſtat item quòd poſito radio = d, & latitudine Zonæ = b, erit ipſa Zona =

$$2bd - \frac{b^3}{3d} - \frac{b^5}{20d^3} - \frac{b^7}{56d^5} \text{ &c.}$$

quare ſi in hoc ipſius Zonæ valore, loco radii = d, ponatur ejus valor, proveniet pro Zonâ ſeu figuræ elemento reſpectu axis B F, Infinita Series infinitarum Serierum, *viz.*

$$L = +2BR - \frac{C^{-2}BRa^2}{} - \frac{C^{-4}BRa^4}{4} - \frac{C^{-6}BRa^6}{8} - \&c.$$
$$- \frac{B^3R^{-1}}{3} - \frac{C^{-2}B^3R^{-1}a^2}{6} - \frac{C^{-4}B^3R^{-1}a^4}{8} - \frac{5C^{-6}B^3R^{-1}a^6}{48} - \&c.$$
$$- \frac{B^5R^{-3}}{20} - \frac{3C^{-2}B^5R^{-3}a^2}{40} - \frac{3C^{-4}B^5R^{-3}a^4}{32} - \frac{7C^{-6}B^5R^{-3}a^6}{64} - \&c.$$
$$- \frac{B^7R^{-5}}{56} - \frac{5C^{-2}B^7R^{-5}a^2}{112} - \frac{5C^{-4}B^7R^{-5}a^4}{64} - \frac{15C^{-6}B^7R^{-5}a^6}{128} - \&c.$$
$$- \&c.$$

Unde *Soliditas Segmenti Spheroidis propositi*

$$= +2BRa - \frac{C^{-2}BRa^3}{3} - \frac{C^{-4}BRa^5}{20} - \frac{C^{-6}BRa^7}{56} - \&c.$$
$$- \frac{B^3R^{-1}a}{3} - \frac{C^{-2}B^3R^{-1}a^3}{18} - \frac{C^{-4}B^3R^{-1}a^5}{40} - \frac{5C^{-6}B^3R^{-1}a^7}{336} - \&c.$$
$$- \frac{B^5R^{-3}a}{20} - \frac{C^{-2}B^5R^{-3}a^3}{40} - \frac{3C^{-4}B^5R^{-3}a^5}{160} - \frac{C^{-6}B^5R^{-3}a^7}{64} - \&c.$$
$$- \frac{B^7R^{-5}a}{56} - \frac{5C^{-2}B^7R^{-5}a^3}{336} - \frac{C^{-4}B^7R^{-5}a^5}{64} - \frac{15C^{-6}B^7R^{-5}a^7}{896} - \&c.$$
$$- \&c.$$

Ut autem Methodi hujus præſtantia ulterius innoteſcat, libet hic ſubjungere inveſtigationem centri gravitatis in Segmento circulari, ad cujus imitationem perfacilè erit cuivis idem in quâvis aliâ figurâ perficere.

Sit circuli Segmentum C D H, cujus radius D A. Ponatur Fig. 23.

$$2\ DA = a$$
$$DB = x$$

Quæritur hujus ſegmenti centrum gravitatis.

Librari intelligatur hoc Segmentum ab E F rectâ, parallelâ ipſi C H, Segmentum in vertice D tangente.

Quoniam hujus figuræ elementum reſpectu axis D B $= 2\sqrt{ax - xx}$. Erit elementi *momentum* reſpectu rectæ E F $= 2x\sqrt{ax - xx} =$

$$2a^{\frac{1}{2}}x^{\frac{3}{2}} - a^{-\frac{1}{2}}x^{\frac{5}{2}} - \frac{1}{4}a^{-\frac{3}{2}}x^{\frac{7}{2}} - \frac{1}{8}a^{-\frac{5}{2}}x^{\frac{9}{2}} - \frac{5}{64}a^{-\frac{7}{2}}x^{\frac{11}{2}}$$

Unde erit *momentum figuræ* C D H, reſpectu rectæ E F $=$

$$\frac{4}{5}a^{\frac{1}{2}}x^{\frac{5}{2}} - \frac{2}{7}a^{-\frac{1}{2}}x^{\frac{7}{2}} - \frac{1}{18}a^{-\frac{3}{2}}x^{\frac{9}{2}} - \frac{1}{44}a^{-\frac{5}{2}}x^{\frac{11}{2}} - \frac{5}{416}a^{-\frac{7}{2}}x^{\frac{13}{2}}$$

Hoc autem diviſum per ipſam aream C D H $=$

$$\frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2}{5}a^{-\frac{1}{2}}x^{\frac{5}{2}} - \frac{1}{14}a^{-\frac{3}{2}}x^{\frac{7}{2}} - \frac{1}{36}a^{-\frac{5}{2}}x^{\frac{9}{2}} - \frac{15}{352}a^{-\frac{7}{2}}x^{\frac{11}{2}} - \&c.$$

Dat *diſtantiam centri gravitatis* ſegmenti C D H à vertice D, in axe D B $=$

$$\frac{3}{5}x - \frac{6}{175}a^{-1}x^2 - \frac{52}{2625}a^{-2}x^3 - \&c.$$

Haud

Haud prorſus diſpari ratione, per hanc Methodum centra percuſſionis & oſcillationis ſemper inveniri poterunt.

Reliquum eſſet ut oſtenderemus, quo pacto quantitatibus relationem elementi figuræ ad elementum baſeos exprimentibus, ſecum invicem vel cum aliis in æquatione involutis, æquatio poſſit reſolvi, adeoque valor elementi figuræ per infinitam ſeriem explicari, in cujus ſingulis terminis, x formæ huic Methodo convenientis exiſteret; quod fit extrahendo æquationum radices in Speciebus, ad modum ferè *Exegetices Numeroſæ Vietææ*, quæ etiam æquationum reſolutio Neceſſaria eſt, cum reciprocè ex datâ areâ quæritur baſis. Utriuſque exemplum unum aut alterum hic ſubjungere viſum eſt.

Fig. 16. Si in *Conchoide Nicomedis* AE, aliter ſupra tractatâ, cujus Norma recta BH, polus C & vertex A æqualiter à normâ diſtantes

Vocentur jam AB = AC a.
BD x.
erit ordinata DE L.

Adeoque ſemper $L^4 + 2aL^3 + x^2L^2 - 2a^2L - a^4 = 0$.
Reſolutâ hâc æquatione erit

$$L = a - \frac{a^{-1}x^2}{8} + \frac{a^{-3}x^4}{128} + \&c.$$

Unde area $BAED = ax - \frac{a^{-1}x^3}{24} + \frac{a^{-3}x^5}{640} + \&c.$

Quo

Quo minor fuerit x, eo pauciores termini areæ approximationi ſufficient.

Eodem modo ſi in hyperbolâ cujus vertex K, aſymptoti AF, AG. Fig. 12.

Ex datis AG M.
AB a.
Areâ hyperbolicâ BCED D.
Quæritur ejuſdem baſis BC x.

Erit ex prædictis in hoc caſu

$$0 = D - \frac{M^2 x}{a} + \frac{M^2 x^2}{2a^2} - \frac{M^2 x^3}{3a^3} + \frac{M^2 x^4}{4a^4} \text{ &c.}$$

& reſolutâ hâc æquatione erit, x ideſt BC =

$$M^{-2} a D + \frac{M^{-4} a D^2}{2} + \frac{M^{-6} a D^3}{6} + \frac{M^{-8} a D^4}{24} \text{ &c.}$$

Non ſecùs in ſolvendo *Kepleri Problemate*, quo quæritur ut, Fig. 24.
dato ſemicirculo AGC, datoque in ipſius diametro puncto D, educatur recta DG ipſum in datâ ratione dividens, ſc: ut ſit ADG ad CDGH ut P ad Q.

Si ponatur radius AB, = R, tertia proportionalis ipſis DB, BC, = B − R, quarta proportionalis ipſis DB, BC, & ſemiperipheriæ AGHC = M. $\frac{PM}{P+Q}$ = a.

AF (talis, ut ex puncto F ductâ rectâ FG, Normali in CA, peripheriam interſecante in G, ſit ducta DG recta quæſita

ſita ſemicirculum in datâ ratione dividens) = x.

Impletis Problematis conditionibus, devenitur in hanc æquationem

$$a = \sqrt{2}\,B\,R^{-\frac{1}{2}}X^{\frac{1}{2}} - \frac{2\sqrt{2}\,R^{-\frac{1}{2}}X^{\frac{3}{2}}}{3\sqrt{2}} + \frac{\sqrt{2}\,B\,R^{-\frac{3}{2}}X^{\frac{3}{2}}}{12} \text{ &c.}$$

Quâ reſolutâ, invenietur AF ideſt

$$x = \frac{R\,a^2}{2\,B^2} + \frac{R\,a^4}{6\,B^5} - \frac{R\,a^4}{26\,B^4} + \frac{R\,a^6}{720\,B^6} \text{ &c.}$$

Quo Minora fuerint diſtantia punctorum D & B, & ratio P ad Q, eo pauciores hujus ſeriei termini in uſum aliquem ſufficient.

Verùm cum hactenus exercitationis limites exceſſerimus explicationem Methodi æquationes in ſeries reſolvendi, in aliam occaſionem rejicimus; niſi id interea præſtitetit (quod equidem libentes auguramur) *Eruditiſſimus D: Walliſius*, quem jàm primùm publico ſcripto moniti ſperamus, inter alia multa præclara ſua aliorumque inventa, Hanc etiàm de æquationum radicibus in ſeries reſolvendis doctrinam, ad mentem *Clariſſimi Summique Analyſtæ D: Iſaaci Neutoni* aliquando enucleaturum; Nobis hâc vice ſatis fuiſſe rati, Methodum tradidiſſe, quâ omnium figuratum areas Geometricè dimetiri licet; (enimvero hanc curvarum per *ſeries infinitas* dimenſionem, non minùs eſſe Geometricam, & in ſuo genere perfectam, utpote ſimpliciſſimam quam rei natura

tura fere poterit, quam illam tranguli vel parabolæ affimare non dubitamus) eamque exemplis benè multis illuſtraſſe, primò quidem figurarum ipſorum etiam veterum ſenſu Geometricam dimenſionem admittentium, deinde *Circuli* aliarumque ejuſdem cum circulo gradus, quales ſunt *Ellipſis*, *Hyperbola*, *Conchois*, *Ciſſois*, *Curva Parabolæ* &c. Deinde etiam illarum quæ compoſitæ magis adhuc ſunt naturæ, hoc eſt quæ vel ſuppoſitâ circuli quadraturâ, ulterius quiddam ad perfectam ſui dimenſionem requirunt, quales ſunt *Curvæ Ellipſeos*, *Hyperbolæ*, &c. Denique ipſam non tantum ad *Curvas* quas vocant *Geometricas*, verum etiam ad *Mechanicas* ut *Cycloidem*, &c Extendiſſe. Lectorem interim admonentes, ad quamvis curvam plurimas exhiberi poſſe ſeries, diverſas pro poſitionum diverſitate, (cujus in *Conchoide* exemplum conſultò dedimus) immò iiſdem licet adhibitis poſitionibus, pro diverſâ elementi reductione (quod in quibuſdam ſpeciatim notavimus) quarum ſimpliciores propoſitoque aptiores eligere oportet: Porro in elemento ad infinitam ſeriem reducendo, eam reductionis viam eſſe vitandam, quâ evenit ut quidam ſeriei terminus infinitam ſuperficiem hyperbolicam inter hyperbolam & aſymptoton infinitè continuatas comprehenſam producat; ipſa tamen aliquando commodè adhibetur, ut in *Longitudine Curvæ Logarithmicæ* inveſtigandâ ſuperiùs factum; Nullum denique hìc

ferè exhiberi exemplum, in quo generale quiddam & pluribus casibus inserviens non contineatur.

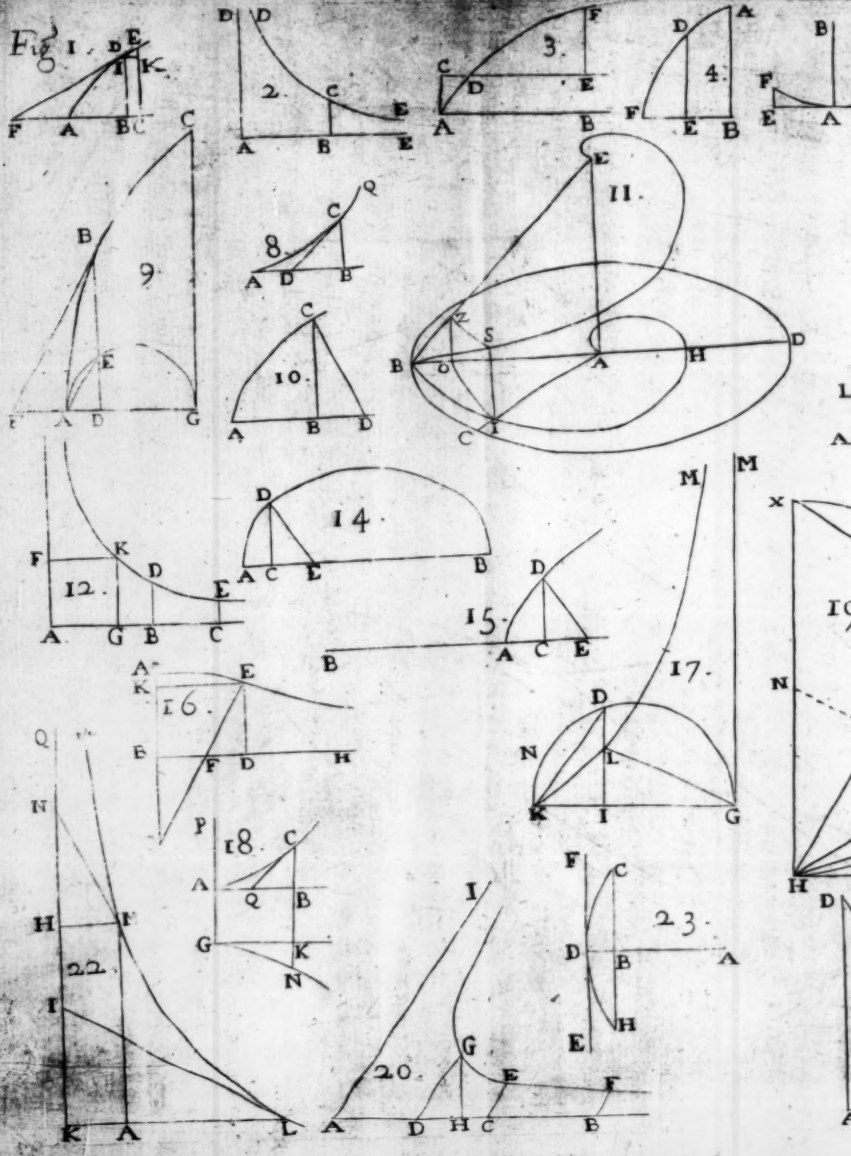
Verum hisce diutiùs immorari non jam vacat, ut neque ostendere, quantùm hæc calculi in plærisque Mathematicis molestias imminuant; quamque simplices cujuslibet problematis constructiones ex serie possunt derivari; neque etiam explicare modum, quo ipsa linea curva &c. loco baseos vel axeos utendum sit, quod aliquando non sine insigni commodo, precipuè in regressibus ab areis ad bases usu venit.

Cæterùm unum idemque Geometriæ promovendæ studium, in causâ fuit, ut & nonnulla quæ Methodo nostræ illustrandæ cum primis erant necessaria, fusiùs prosecuti fuerimus; & alia non pauca quæ lectorum ingeniis exercendis & excolendis inservire poterunt, intento solum digito sublindicaverimus; aliqua denique omni ferè explicatione, nisi quæ ab exemplis emanat, destituta tantùm non omiserimus, dilatâ in aliud tempus eorum tractatione, ne aliis pari hanc scientiam augendi desiderio occupatis, occasionem benè merendi de civitate Mathematicorum præripere velle, hoc est de eâ malè mereri videremur.

FINIS.

# ERRATA.

Pag: 1. l: 18. pro anarlyticè leg: analyticè. pag: 5. lin: 2. pro $x^{-\frac{4}{5}}$ leg: $A^{-\frac{4}{5}}$. pag: 13. lin: penult: pro 2AE leg: L. Unde longitudo Curvæ AB $=$ $2D^{\frac{1}{2}}x^{\frac{1}{2}} =$ 2AE. Pag: 18: lin: ult: pro ultorius leg: ulteriùs. Pag: 22. lin: 5: pro $\frac{7RC^{-12}x^{13}}{2816}$ leg: $\frac{7RC^{-10}x^{11}}{2816}$. pag: 31: lin: 3: pro $x^3$ leg. $\frac{1}{4}x^2$. pag: 40: lin: 1: pro $B^5$ leg: $B^4$. pag: 48: lin: 6: pro 26 $B^4$ leg: 24 $B^4$. præter variis in locis æ loco e.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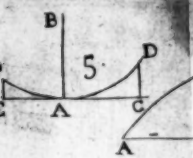
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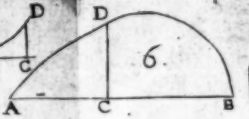
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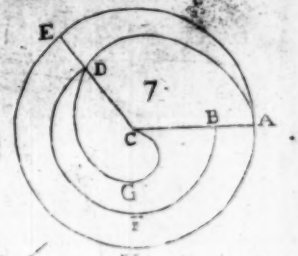
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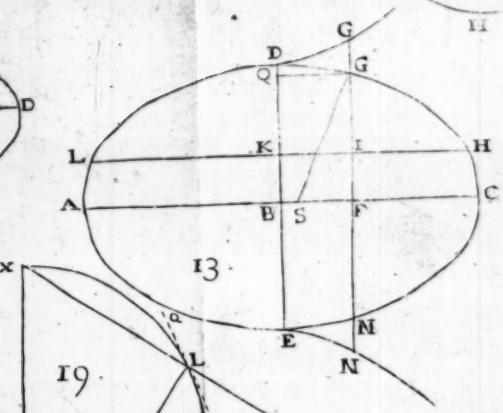
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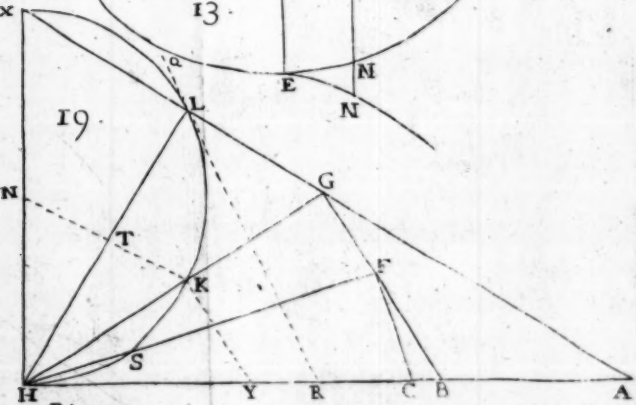
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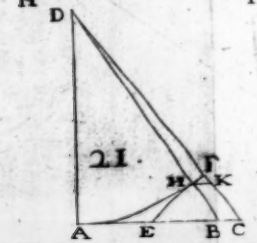
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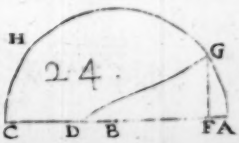
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